5-4 Rational Functions

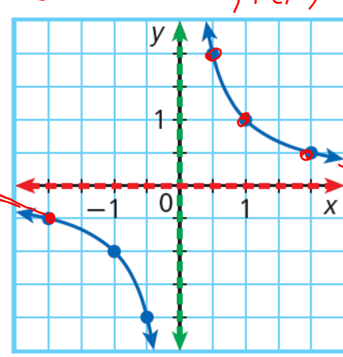
A **rational function** is a function whose rule can be written as a ratio of two polynomials.

The parent rational function is $f(x) = \frac{1}{x}$. Its graph is a *hyperbola*, which has two separate branches. You will learn more about hyperbolas later in this course.

5-4 Rational Functions

as $x \to \infty$, $f(x) \to 0$
as $x \to -\infty$, $f(x) \to 0$

The function $f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.



x	$\frac{1}{x}$
-2	$-\frac{1}{2}$
0	undefined
1	1
2	$\frac{1}{2}$
$\frac{1}{2}$	2

5-4 Rational Functions

The rational function $f(x) = \frac{1}{x}$ can be transformed by using methods similar to those used to transform other types of functions.

$|a|$ ⟶ vertical stretch or compression factor
$a < 0$ ⟶ reflection across the x-axis

$$f(x) = \frac{a}{x - h} + k$$

k ⟶ vertical translation
down for $k < 0$; up for $k > 0$

h ⟶ horizontal translation
left for $h < 0$; right for $h > 0$

5-4 Rational Functions

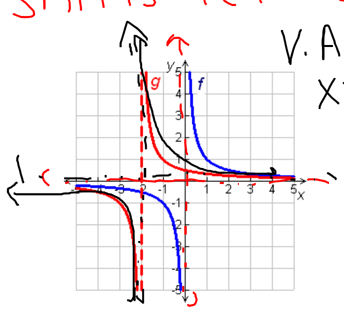
Example 1: Transforming Rational Functions

Using the graph of $f(x) = \frac{1}{x}$ as a guide, describe the transformation and graph each function.

A. $g(x) = \frac{1}{x + 2}$

Shifts left 2

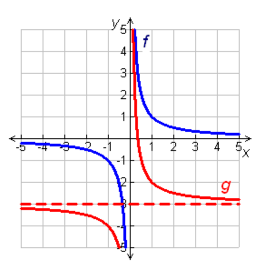
V.A.
x=-2



B. $g(x) = \frac{1}{x} - 3$

down 3

HA:
y=-3



5-4 Rational Functions

Check It Out! Example 1

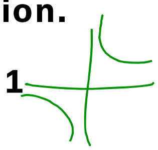
Using the graph of $f(x) = \frac{1}{x}$ as a guide, describe the transformation and graph each function.

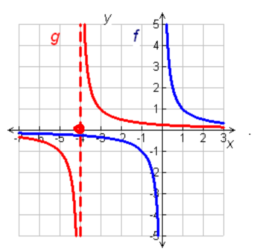
a. $g(x) = \dfrac{1}{x + 4}$

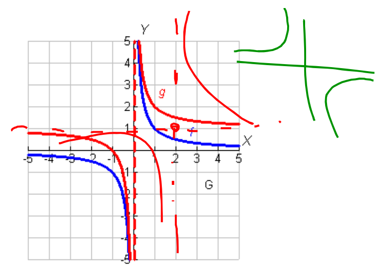
$(h = -4$

$k = 0$

b. $g(x) = \dfrac{1}{x-2} + 1$







5-4 Rational Functions

The values of *h* and *k* affect the locations of the asymptotes, the domain, and the range of rational functions whose graphs are hyperbolas.

Rational Functions

For a rational function of the form $\boxed{f(x) = \dfrac{a}{x - h} + k}$,

- the graph is a hyperbola. ✓
- there is a vertical asymptote at the line $x = h$, and the domain is $\{x \mid x \neq h\}$ $(-\infty, h) \cup (h, \infty)$
- there is a horizontal asymptote at the line $y = k$, and the range is $\{y \mid y \neq k\}$. $(-\infty, k) \cup (k, \infty)$

5-4 Rational Functions

Example 2: Determining Properties of Hyperbolas

Identify the asymptotes, domain, and range of the function $g(x) = \frac{1}{x + 3} - 2$**.**

Vertical asymptote: $x = -3$

Domain: $x \neq -3$

Horizontal asymptote: $y = -2$

Range: $y \neq -2$

Check Graph the function on a graphing calculator.

5-4 Rational Functions

Check It Out! Example 2

Identify the asymptotes, domain, and range of the function $g(x) = \frac{1}{x - 3} - 5$**.**

Vertical asymptote:

Domain:

Horizontal asymptote:

Range:

Check Graph the function on a graphing calculator.

5-4 Rational Functions

A **discontinuous function** is a function whose graph has one or more gaps or breaks. The hyperbola graphed in Example 2 and many other rational functions are discontinuous functions.

A **continuous function** is a function whose graph has no gaps or breaks. The functions you have studied before this, including linear, quadratic, and polynomial are continuous functions.

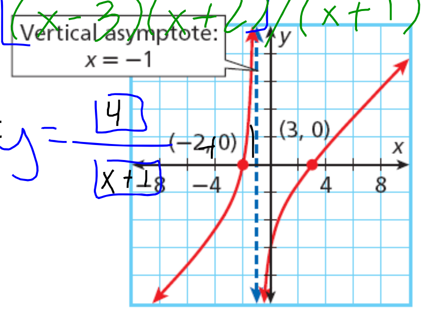
5-4 Rational Functions

The graphs of some rational functions are not hyperbolas. Consider the rational function $f(x) = \frac{(x-3)(x+2)}{x+1}$ and its graph.

$y = \boxed{(x-3)(x+2)}/(x+1)$

The numerator of this function is 0 when $x = 3$ or $x = -2$. Therefore, the function has x-intercepts at -2 and 3. The denominator of this function is 0 when $x = -1$. As a result, the graph of the function has a vertical asymptote at the line $x = -1$.

$f(x) = \frac{4}{x+1}$ $y = \frac{\boxed{4}}{\boxed{x+1}}$



5-4 Rational Functions

Zeros and Vertical Asymptotes | **Rational Functions**

If $\boxed{f(x) = \dfrac{p(x)}{q(x)}}$, where p and q are polynomial functions in standard form with no common factors other than 1, then the function f has

- zeros at each real value of x for which $p(x) = 0$. numerator $=0$
- a vertical asymptote at each real value of x for which $q(x) = 0$.

denominator $=0$

5-4 Rational Functions

Example 3: Graphing Rational Functions with Vertical Asymptotes

Identify the zeros and vertical asymptotes of $f(x) = \dfrac{(x^2 + 3x - 4)}{x + 3}$.

$f(x) = \dfrac{(x+4)(x-1)}{x+3}$

Step 1 Find the zeros and vertical asymptotes.

$f(x) = \dfrac{(x + 4)(x - 1)}{x + 3}$ — *Factor the numerator.*

Zeros: $(-4, 0)$, $(1, 0)$

Zeros: –4 and 1 — *The numerator is 0 when $x = -4$ or $x = 1$.*

VA: $x = -3$

Vertical asymptote: $x = -3$ — *The denominator is 0 when $x = -3$.*

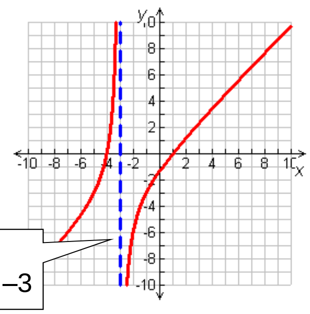
5-4 Rational Functions

Example 3 Continued

Identify the zeros and vertical asymptotes of $f(x) = \frac{(x^2 + 3x - 4)}{x + 3}$.

Step 2 Graph the function.

Plot the zeros and draw the asymptote. Then make a table of values to fill in missing points.



x							
y							

5-4 Rational Functions

Check It Out! Example 3

Identify the zeros and vertical asymptotes of $f(x) = \frac{(x^2 + 7x + 6)}{x + 3}$.

Step 1 Find the zeros and vertical asymptotes.

$f(x) = \frac{(x + 6)(x + 1)}{x + 3}$ *Factor the numerator.*

Zeros: –6 and –1 *The numerator is 0 when $x = -6$ or $x = -1$.*

Vertical asymptote: $x = -3$ *The denominator is 0 when $x = -3$.*

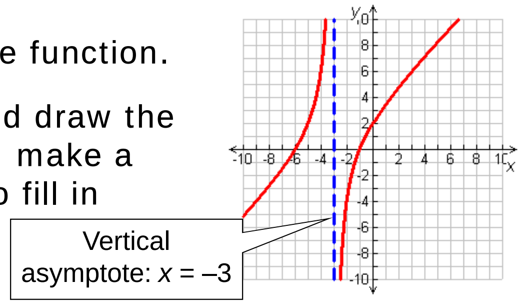
5-4 Rational Functions

Check It Out! Example 3 Continued

Identify the zeros and vertical asymptotes of $f(x) = \frac{(x^2 + 7x + 6)}{x + 3}$.

Step 2 Graph the function.

Plot the zeros and draw the asymptote. Then make a table of values to fill in missing points.



x							
y							

5-4 Rational Functions

Warm Up (Post 5.4, Part I)

1. Using the graph of $f(x) = \frac{1}{x}$ as a guide, describe the transformation and graph the function $g(x) = \frac{1}{x - 4}$.

2. Identify the asymptotes, domain, and range of the function $g(x) = \frac{5}{x - 1} + 2$.

5-4 Rational Functions

Some rational functions, including those whose graphs are hyperbolas, have a horizontal asymptote. The existence and location of a horizontal asymptote depends on the degrees of the polynomials that make up the rational function.

Note that the graph of a rational function can sometimes cross a horizontal asymptote. However, the graph will approach the asymptote when $|x|$ is large.

5-4 Rational Functions

Horizontal Asymptotes | **Rational Functions**

Let $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions in standard form with no common factors other than 1. The graph of f has at most one horizontal asymptote.

- If degree of $p >$ degree of q, there is no horizontal asymptote.
- If degree of $p <$ degree of q, the horizontal asymptote is the line $y = 0$.
- If degree of $p =$ degree of q, the horizontal asymptote is the line

$$y = \frac{\text{leading coefficient of } p}{\text{leading coefficient of } q}.$$

5-4 Rational Functions

Example 4A: Graphing Rational Functions with Vertical and Horizontal Asymptotes

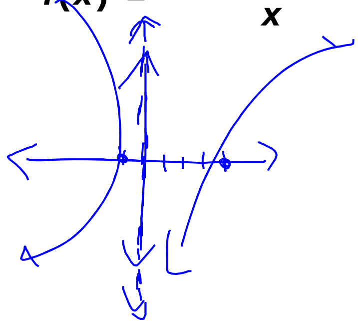
Identify the zeros and asymptotes of the function. Then graph.

$$f(x) = \frac{x^2 - 3x - 4}{x} = \frac{(x-4)(x+1)}{x}$$

Zeros: $(4,0), (-1,0)$

V.A. $x=0$

H.A. none



5-4 Rational Functions

Example 4B: Graphing Rational Functions with Vertical and Horizontal Asymptotes

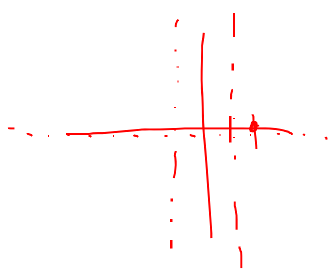
Identify the zeros and asymptotes of the function. Then graph.

$$f(x) = \frac{x - 2}{x^2 - 1} = \frac{x-2}{(x+1)(x-1)}$$

Zeros: $(2,0)$

V.A. $x=1$, $x=-1$

H.A. $y=0$



5-4 Rational Functions

Example 4C: Graphing Rational Functions with Vertical and Horizontal Asymptotes

Identify the zeros and asymptotes of the function. Then graph.

$$f(x) = \frac{4x - 12}{x - 1} = \frac{4(x-3)}{x-1}$$

Zeros: (3,0)

VA: x = 1

HA: y = 4

5-4 Rational Functions

Check It Out! Example 4a

Identify the zeros and asymptotes of the function. Then graph.

$$f(x) = \frac{x^2 + 2x - 15}{x - 1}$$

5-4 Rational Functions

Check It Out! Example 4b

Identify the zeros and asymptotes of the function. Then graph.

$$f(x) = \frac{x - 2}{x^2 + x}$$

5-4 Rational Functions

Check It Out! Example 4c

Identify the zeros and asymptotes of the function. Then graph.

$$f(x) = \frac{3x^2 + x}{x^2 - 9}$$

5-4 Rational Functions

In some cases, both the numerator and the denominator of a rational function will equal 0 for a particular value of *x*. As a result, the function will be undefined at this *x*-value. If this is the case, the graph of the function may have a *hole*. A **hole** is an omitted point in a graph.

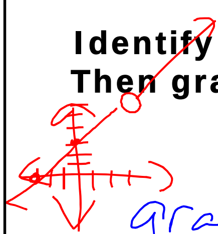
Holes in Graphs | **Rational Functions**

If a rational function has the same factor $x - b$ in both the numerator and the denominator, then there is a hole in the graph at the point where $x = b$, unless the line $x = b$ is a vertical asymptote.

5-4 Rational Functions

Example 5: Graphing Rational Functions with Holes

Identify holes in the graph of $f(x) = \dfrac{x^2 - 9}{x - 3}$. Then graph.



$$f(x) = \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$

graph of f(x) resembles
$y = x+3$ but there's a hole
at $x = 3$ (at point (3,6))

Check It Out! Example 5

Identify holes in the graph of $f(x) = \dfrac{x^2 + x - 6}{x - 2}$. Then graph.