3-5 Finding Real Roots of Polynomial Equations
and
3-6 Fundamental Theorem of Algebra

Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

Every polynomial function of degree $n \geq 1$ has at least one zero, where a zero may be a complex number.

Corollary: Every polynomial function of degree $n \geq 1$ has exactly n zeros, including multiplicities.

Examples: Solve each polynomial equation by factoring.

1. $x^3 - 2x^2 - 25x = -50$

$x^3 - 2x^2 - 25x + 50 = 0$

$x^2(x-2) - 25(x-2) = 0$

$(x^2-25)(x-2) = 0$

$(x+5)(x-5)(x-2) = 0$

$x = 5, -5, 2$

2. $4x^6 + 4x^5 - 24x^4 = 0$

$4x^4(x^2 + x - 6) = 0$

$4x^4(x+3)(x-2) = 0$

$4x \cdot x \cdot x \cdot x(x+3)(x-2) = 0$

$4x^4 = 0$

$x^4 = 0$

$x = 0$

$x = 0, -3, 2$

3. $x^4 + 25 = 26x^2$

$x^4 - 26x^2 + 25 = 0$

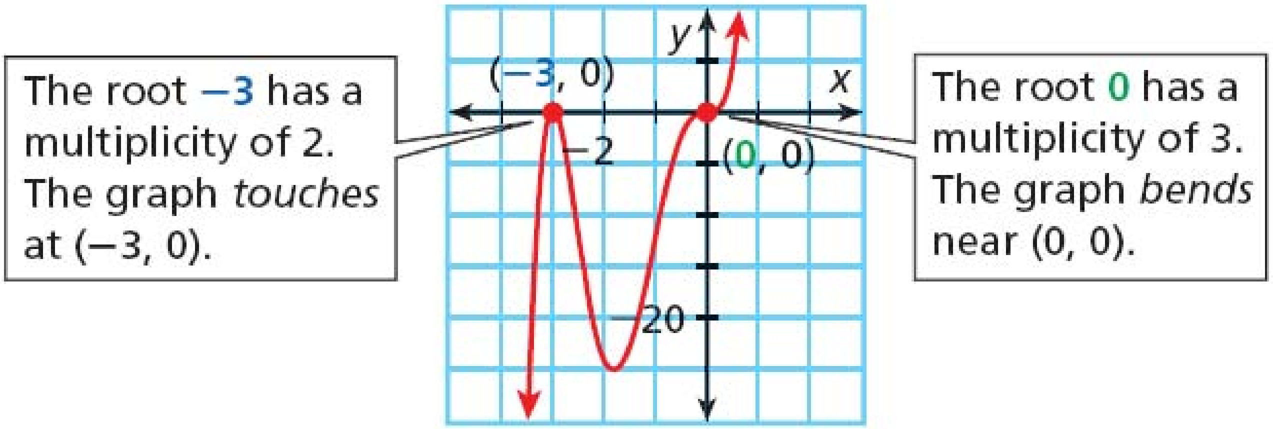
$(x^2-25)(x^2-1) = 0$

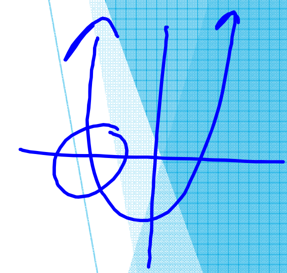
$(x+5)(x-5)(x+1)(x-1) = 0$

$x = \pm 5, \pm 1$

Sometimes a polynomial equation has a factor that appears more than once. This creates a *multiple root*.

- The **multiplicity** of root r is the number of times that $x - r$ is a factor of $P(x)$.
- When a real root has even multiplicity, the graph of $y = P(x)$ touches the x-axis but does not cross it.
- When a real root has odd multiplicity greater than 1, the graph "bends" as it crosses the x-axis.





You cannot always determine the multiplicity of a root from a graph. It is easiest to determine multiplicity when the polynomial is in factored form.

Examples: Identify the roots of each equation. State the multiplicity of each root.

4. $x^3 + 6x^2 + 12x + 8 = 0$

$(x+2)^3 = 0$

$$\begin{array}{r|rrr|r} -2 & 1 & 6 & 12 & 8 \\ & & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$x = -2$, multiplicity of 3

$x^2 + 4x + 4 = (x+2)(x+2)$

5. $x^4 + 8x^3 + 18x^2 - 27 = 0$ $\rightarrow (x+3)^3(x-1) = 0$

$$\begin{array}{r|rrrr|r} -3 & 1 & 8 & 18 & 0 & -27 \\ & & -3 & -15 & -9 & 27 \\ \hline & 1 & 5 & 3 & -9 & 0 \end{array}$$

$$\begin{array}{r|rrr|r} 1 & 1 & 5 & 3 & -9 \\ & & 1 & 6 & 9 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$x^2 + 6x + 9 = (x+3)^2$

$x = -3$, multiplicity of 3

$x = 1$, multiplicity of 1

Examples: Solve each equation by factoring. State the multiplicity of each root. $x^2 - 5 = 0$ $x^2 = 5$ $x = \pm\sqrt{5}$

6. $x^3 + 6x^2 - 5x - 30 = 0$

$x^2(x+6) - 5(x+6) = 0$

$(x^2 - 5)(x+6) = 0$

$(x+\sqrt{5})(x-\sqrt{5})(x+6) = 0$

$x = \sqrt{5}$, multiplicity 1

$x = -\sqrt{5}$, "

$x = 6$, "

7. $2x^5 + 12x^4 + 16x^3 - 12x^2 - 18x = 0$

$2x(x^4 + 6x^3 + 8x^2 - 6x - 9) = 0$

$2x(x+3)^2(x+1)^1(x-1)^1 = 0$

$x = 0$ $x = -3$ $x = 1$ $x = -1$

mult:

$x = 0, 1, -1$ each with mult of 1

$x = -3$, mult. of 2

Rational Root Theorem

If the polynomial $P(x)$ has integer coefficients, then every rational root of the polynomial equation $P(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term of $P(x)$ and q is a factor of the leading coefficient of $P(x)$.

Irrational Root Theorem

If the polynomial $P(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the polynomial equation $P(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $P(x) = 0$.

Complex Conjugate Root Theorem

If $\boldsymbol{a + bi}$ is a root of a polynomial equation with real-number coefficients, then $\boldsymbol{a - bi}$ is also a root.

Let's put it all together. First, here is an overview of the steps for finding ALL roots of a polynomial equation.

Solving Polynomial Equations

1. **Use the Rational Root Theorem to identify all possible rational roots.**

2. **Graph the polynomial equation to find the real roots AND / OR test the possible rational roots to find at least one that is actually a root.**

3. **Use the identified root(s) to factor the polynomial completely.**

4. **Solve and state ALL roots of the original equation.**

Examples: Solve each equation by finding all roots.

1. $4x^4 - 21x^3 + 18x^2 + 19x - 6 = 0$

① List possible rational zeros: $\dfrac{\text{factors of } -6}{\text{factors of } 4} = \dfrac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2, \pm 4}$

$= \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 2, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm 6$

② Identify 2 rational zeros: $-3/4$, 2

③ Divide

$$\begin{array}{r|rrrrr} 2 & 4 & -21 & 18 & 19 & -6 \\ & & 8 & -26 & -16 & 6 \\ \hline -3/4 & 4 & -13 & -8 & 3 & 0 \\ & & -3 & 12 & -3 & \\ \hline & 4 & -16 & 4 & 0 & \end{array}$$

④ Solve

$4x^2 - 16x + 4 = 0$

$x^2 - 4x + 1 = 0$

$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2}$

$x =$

$$x^2 - 4x + 1 = 0$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2}$$

$$x = \frac{4 \pm \sqrt{12}}{2}$$

$$x = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

5 Identify all solutions

$$x = -3/4,\ 2,\ 2 \pm \sqrt{3}$$