Warm Up

Use your calculator to determine if the following statements are true.

1. $\sin(390° + 120°) = \sin 390° + \sin 120°$

2. $\cos\left(\frac{\pi}{6} - \frac{\pi}{3}\right) = \cos\frac{\pi}{6} - \cos\frac{\pi}{3}$

5.4 Sum and Difference Formulas

Sum and Difference Formulas

$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

$\tan(u \pm v) = \dfrac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$

These formulas are used to find *exact* trigonometric values using the basic trigonometric values (of special angles) we already know.

Example #1 **Find the exact values of sine, cosine, and tangent of 105°.**

$\sin 105° = \sin(60° + 45°)$

$= \sin 60° \cos 45° + \cos 60° \sin 45°$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$

$\cos 105° = \cos(60° + 45°)$

$= \cos 60° \cos 45° - \sin 60° \sin 45°$

$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$

$\tan 105° = \tan(60° + 45°)$

$= \dfrac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°} = \dfrac{(\sqrt{3}+1)}{(1-\sqrt{3})} \cdot \dfrac{(1+\sqrt{3})}{(1+\sqrt{3})} =$

$= \dfrac{\sqrt{3} + 3 + 1 + \sqrt{3}}{1 - 3} = \dfrac{4+2\sqrt{3}}{-2}$

$\tan 105° = -2 - \sqrt{3}$

Example #2 **Find the exact values of sine, cosine, and tangent of $\frac{\pi}{12}$.**

$\frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{2} = \frac{\pi}{12}$ $\frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{12}$

$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{6}+\sqrt{2}}{4}$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{\left(1 - \frac{\sqrt{3}}{3}\right)\left(1 - \frac{\sqrt{3}}{3}\right)}{\left(1 + \frac{\sqrt{3}}{3}\right)\left(1 - \frac{\sqrt{3}}{3}\right)}$

$= \frac{1 - \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + \frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{4}{\cancel{3}} - \frac{2\sqrt{3}}{\cancel{3}}}{\frac{2}{\cancel{3}}}$

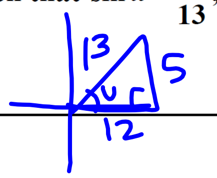
$\frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$

Example #3 **Write $\sin 3.5\cos 1.2 - \cos 3.5\sin 1.2$ as the sine, cosine or tangent of an angle.**

$\sin(3.5 - 1.2) = \sin 2.3$

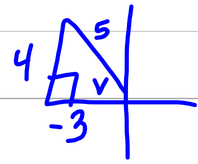
Example #4 **Find the exact value of** $\cos(u+v)$ **given that** $\sin u = \frac{5}{13}$,

$0 < u < \frac{\pi}{2}$, $\cos v = -\frac{3}{5}$, and $\frac{\pi}{2} < v < \pi$



$\cos(u+v) = \cos u \cos v - \sin u \sin v$

$= \left(\frac{12}{13}\right)\left(-\frac{3}{5}\right) - \left(\frac{5}{13}\right)\left(\frac{4}{5}\right)$

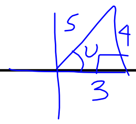


$-\frac{36}{65} - \frac{20}{65}$

$\cos(u+v) = -56/65$

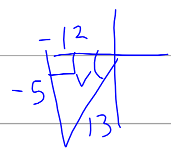
Example #5 **Find the exact value of** $\sin(u+v)$ **given that** $\sin u = \frac{4}{5}$,

$0 < u < \frac{\pi}{2}$, $\cos v = -\frac{12}{13}$, and $\pi < v < \frac{3\pi}{2}$



$\sin(u+v) = \sin u \cos v + \cos u \sin v$

$= \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)$



$= -\frac{48}{65} + -\frac{15}{65} = \frac{-63}{65}$

Example #6 **Write the trigonometric expression as an algebraic expression:**

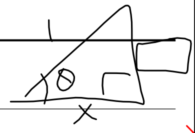
$$\cos(\arctan 1 + \arccos x)$$

$x^2 + b^2 = 1^2$

$x^2 + b^2 = 1$

$b^2 = 1 - x^2$

$b = \sqrt{1-x^2}$



$\cos(\arctan 1 + \arccos x)$

$= \cos(\arctan 1)\cos(\arccos x) - \sin(\arctan 1)\sin(\arccos x)$

$= \cos\left(\frac{\pi}{4}\right) \quad x \quad - \sin\frac{\pi}{4}$

$\left(\frac{\sqrt{2}}{2}\right)(x) - \left(\frac{\sqrt{2}}{2}\right)\left(\sqrt{1-x^2}\right)$

$\frac{x\sqrt{2}}{2} - \frac{\sqrt{2-2x^2}}{2} = \frac{x\sqrt{2} - \sqrt{2-2x^2}}{2}$